Boltzmann Distribution and Temperature of Stock Markets

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We point out that the high-frequency fluctuations of of S&P 500 and NASDAQ 100 indices show Boltzmann statistics over a wide range of positive as well as negative returns, thus allowing us to define a market temperature for either sign. With increasing time the sharp Boltzmann peak broadens into a Gaussian whose volatility σ measured in 1/min is related to the temperature T by $T = \sigma/\sqrt{2}$. Plots over the years 1990–2006 show that the arrival of the 2000 crash was preceded by a dramatic increase in market temperature.

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It is by now well-known that financial data do no display Gaussian distributions [1-5]. Most importantly, the tails of the distributions are power-like, since large fluctuations are much more frequent than in a Gaussian distribution. This is of great importance for financial institutions who want to estimate the risk of market crashes. point out

In this note we would like to focus on the opposite regime of the most frequent events near the peak of the distribution. We want to point out that the highest-frequency returns x of NASDAQ 100 and S&P 500 indices have a special property: they display a Boltzmann distribution for positive as well as negative x, as long as the probability is rather large. The data are fitted by the Boltzmann distribution

$$\tilde{B}(x) = \frac{1}{2T} e^{-|x|/T}. \tag{1}$$

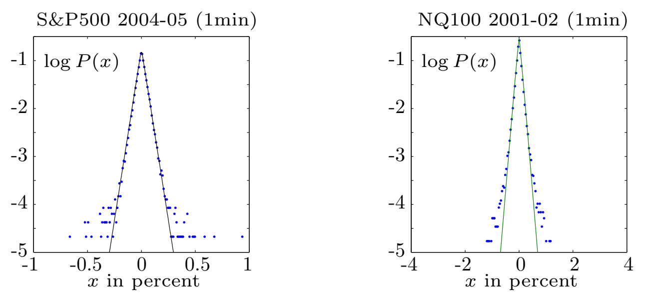


FIG. 1: Boltzmann distribution of S&P 500 and NASDAQ 100 high-frequency returns recorded by the minute.

We can see in Fig. 1, that only a very small set of rare events of large $|x|$ does not follow the Boltzmann law, but displays heavy tails. This allows us to assign a temperature to the stock markets. In principle, there are different temperature $T_\pm$ for positive and negative returns, but to a good approximations we may equate both $T \approx T_+ \approx T_-$. The temperature T depends on the selection of stocks and changes only very slowly with

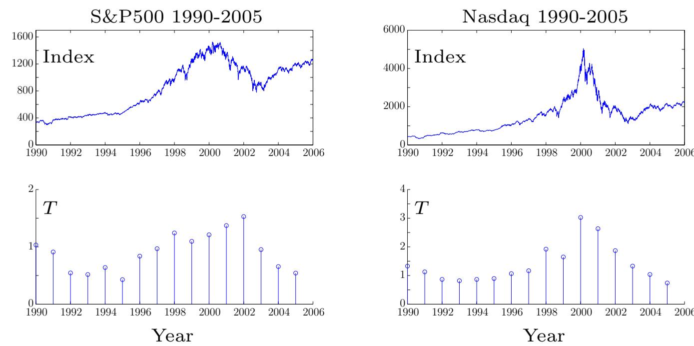


FIG. 2: Market temperatures of S&P 500 and NASDAQ indices from 1990 to 2006. The crash of NASDAQ in the year 2000 occurred at the maximal temperatures $T_{\rm NASDAQ} \approx 3$.

the general economic and political environment. Near a crash is reaches maximal values, as shown in Fig. 2.

For lower frequencies, the distribution becomes more and more Gaussian, as required by the *central limiting theorem* of statistical mechanics which states that the convolution of infinitely many arbitrary distribution functions of finite width always approaches a Gaussian distribution. This is illustrated by the the weekly data of the two indexes in Fig. 3.

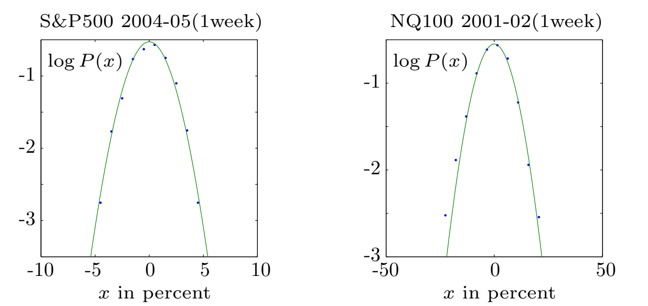
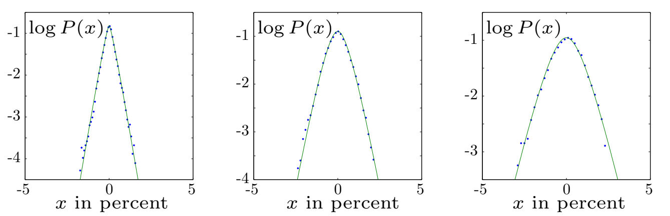


FIG. 3: Gaussian distributions of S&P 500 and NASDAQ 100 data recorded by the week.

The transition from Boltzmann to Gaussian distributions is shown for the S&P 500 index in Fig. 4.

The convergence to a Gaussian distribution is in contrast to the pure Lévy distribution of infinite width where the falloff remains power-like at large distances for any data frequency. The same thing will happen here for the extremely rare events which lie outside of the Boltzmann regime.



(a) (b) (c)

FIG. 4: Fits of convolution of Boltzmann distribution to S&P 500 data taken in intervals of 1 hour, 4 hours, and 1 day, respectively.

The time dependence of the distribution is found in the usual way. We calculate the Fourier transform of $B(x)$:

$$B(p) = \int_{-\infty}^{\infty} dx\, e^{ipx} \frac{1}{2T} e^{-|x|/T} = \frac{1}{1+(Tp)^2}, \tag{2}$$

and identify the Hamiltonian as

$$H(p) = \log[1 + (Tp)^2]. \tag{3}$$

This has only even cumulants ($n = 2, 4, \dots$):

$$c_n = -i^n H^{(n)}(0) = 2i^n (-1)^{n/2} T^n (n-1)!. \tag{4}$$

As a function of time, the distribution widens as fol-

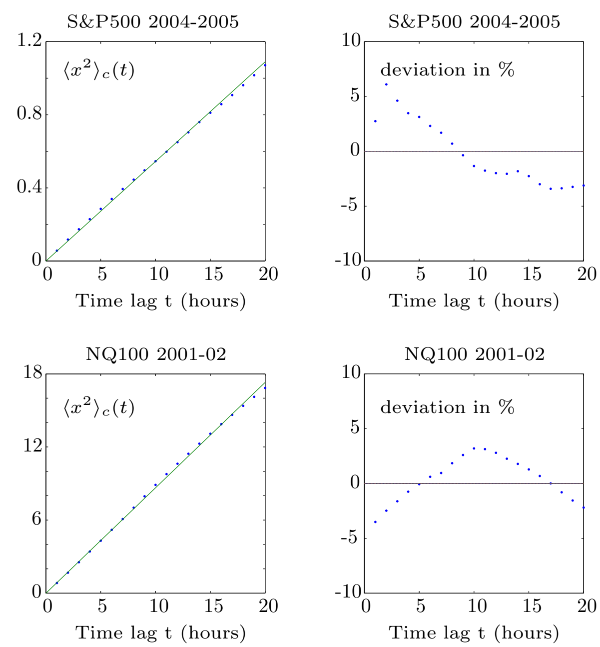


FIG. 5: Variance of S&P 500 and NASDAQ 100 indices as a function of time. The right-hand side amplifies the small relative deviation from the linear shape in percent.

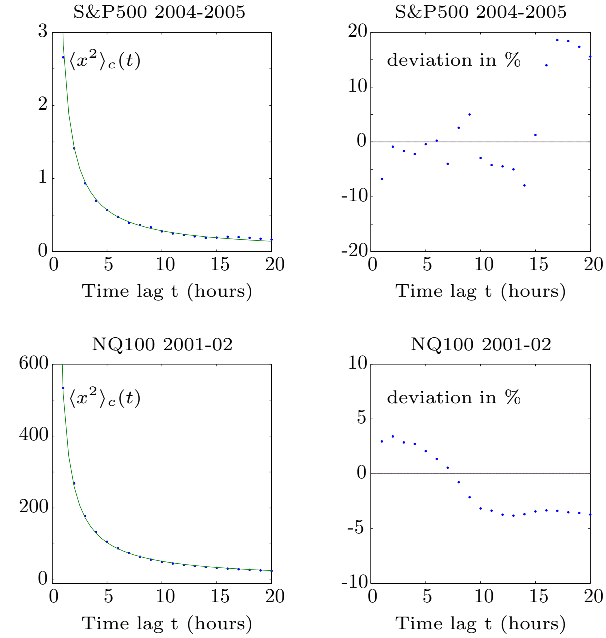


FIG. 6: Kurtosis of S&P 500 and NASDAQ 100 indices as a function of time. The right-hand side shows the relative deviation from the $1/t$ behavior in percent.

lows:

$$\begin{aligned} \tilde{B}(x;t) &= \int_{-\infty}^{\infty} \frac{dp}{2\pi} e^{ipx - tH(p)} \qquad (5) \\ &= \frac{1}{T\sqrt{\pi}\Gamma(t)} \left(\frac{|x|}{2T}\right)^{t-1/2} K_{t-1/2}(|x|/T). \end{aligned}$$

where t is measured in minutes. For $t = 1$, this reduces to (1).

The variance and kurtosis of this distribution increases linearly in time as

$$\sigma^2(t) \equiv \langle x^2 \rangle_c(t) = t\sigma^2 = 2T^2 t,\ \kappa(t) \equiv \frac{\langle x^4 \rangle_c(t)}{\langle x^2 \rangle_c^2(t)} - 3 = \frac{3}{t}. \tag{6}$$

These quantities are plotted in Figs. 5 and Fig. 6.

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[1] R.N. Mantegna and H.E. Stanley *Stochastic Process with Ultraslow Convergence to a Gaussian:The Truncated Lévy Flight*, Phys.Rev.Letters **73**, 2949 (1994); *Scaling Behaviour in the Dynamics of an Economic Index*, Nature **376**, 46 (1995); *Econophysics: Scaling and Its Breakdown in Finance*, J.Stat.Phys **89**, 469 (1997); *An Introduction to Econophysics: Correlations and Complexity in Finance*, Cambridge University Press, 2000.
[2] J.-P. Bouchaud and R. Cont *Elements for a Theory of Financial Risk*, Physica A **263**, 415 (1999); *A Langevin*

Approach to Stock Market Fluctuations and Crashes, Eur.Phys.J.B **6**, 543 (1998)

[3] L. Laloux,M. Potters,R. Cont,J.-P. Aguilar and J.-P. Bouchaud *Are Financial Crashes Predictable?*,Europhys.Lett. **45**, 1 (1999)

[4] J.-P. Bouchaud and M. Potters, *Theory of Financial Risks: From Statistical Physics to Risk Management*, Cambridge University Press, 2000.

[5] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets*, World Scientific, Singapore 2004, Third extended edition, pp. 1–1450 (http://www.physik.fu-berlin.de/~kleinert/b5).

[6] W.Paul and J.Baschnagel, *Stochastic Processes: From Physics to Finance*, Springer, 2000.

[7] J. Voigt, *The Statistical Mechanics of Financial Markets*, Springer, Berlin, 2001.

[9] A.C.Silva and V.M.Yakovenko *Comparison between the probability distribution of returns in the Heston model and empirical data for stock indexes*, physica A **324**, 303 (2003).

[9] A.C.Silva,R.E.Prange and V.M.Yakovenko *Exponential distribution of financial returns at mesoscopic time lags:a new stylized fact*, physica A **344**, 227 (2004).

[10] H. Kleinert, *Stochastic Calculus for Assets with Non-Gaussian Price Fluctuations*, Physica A **311**, 538 (2002).

[11] H. Kleinert, *Option Pricing from Path Integral for Non-Gaussian Fluctuations. Natural Martingale and Application to Truncated Lévy Distributions*, Physica A **312**, 217 (2002).

[12] F. Black and M. Scholes, *The Pricing of Options and Corporate Liabilities*,J. Pol. Economy **81**, 637 (1973).

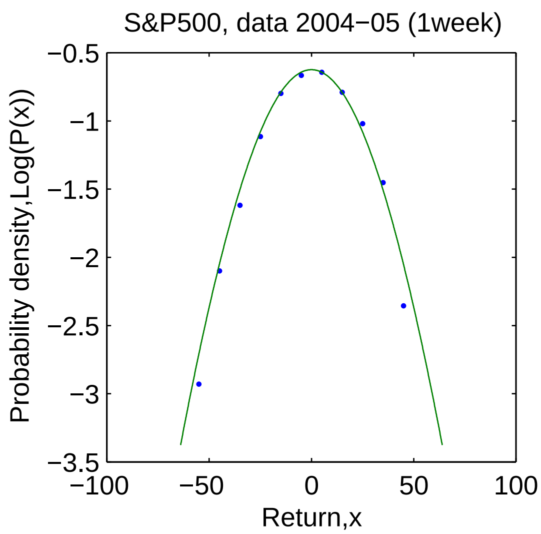
[13] R.C. Merton, *Theory of Rational Option Pricing*, Bell J. Econ. Management Sci. **4**, 141 (1973).

[14] C. Tsallis, J. Stat. Phys. **52**, 479 (1988); E.M.F. Curado and C. Tsallis, J. Phys. A **24**, L69 (1991); 3187 (1991); A **25**, 1019 (1992).

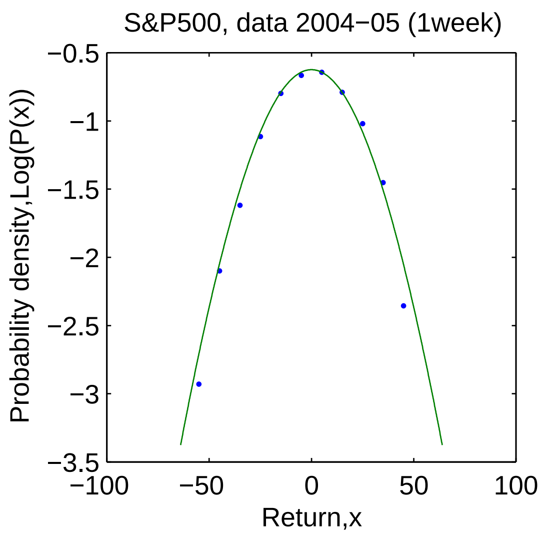
[15] C. Tsallis, C. Anteneodo, L. Borland, R. Osorio, *Nonextensive Statistical Mechanics and Economics*, Physica A **324**, 89 (2003) (cond-mat/030130).

[16] Yahoo Finance `http://finance.yahoo.com`. To download data, enter in the symbol box: `^DJI`, and then click on the link: *Download Spreadsheet*.

[17] `http://www.tickdata.com`.



S&P500, data 2004−05 (1week)
Probability density,Log(P(x))
Return,x
−0.5
−1
−1.5
−2
−2.5
−3
−3.5
−100
−50
0
50
100


S&P500, data 2004−05 (1week)
Probability density,Log(P(x))
Return,x
−0.5
−1
−1.5
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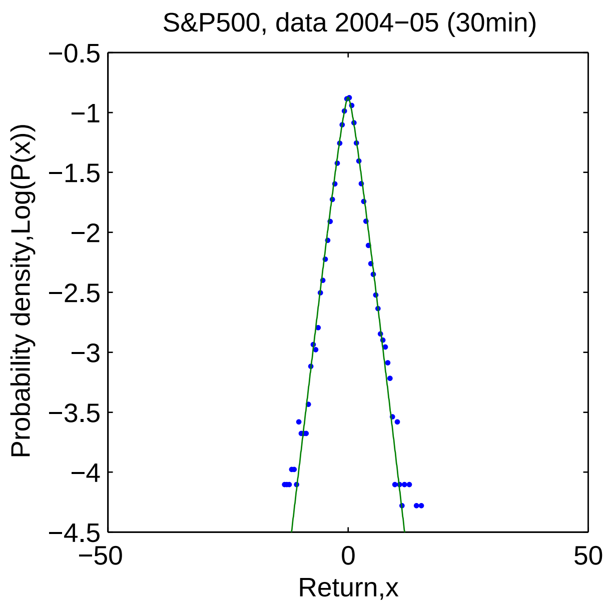
−2.5
−3
−3.5
−100
−50
0
50
100


S&P500, data 2004−05 (30min)
Probability density, Log(P(x))
Return, x
−0.5
−1
−1.5
−2
−2.5
−3
−3.5
−4
−4.5
−50
0
50